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Q) If $P(x) = \frac{x}{10}$, $x = 0, 1, 2, 3$

$= 0$, else where

Find the prob. of

i) $X = 2$ or 3

ii) $P\left[\frac{1}{3} < x < \frac{7}{2} \mid x > 1\right]$

Soln $P(X = 2) = \frac{2}{10}$

$P(X = 3) = \frac{3}{10}$

$P(X = 2 \text{ or } 3) = \frac{2}{10} + \frac{3}{10} = \frac{5}{10} = \frac{1}{2}$

ii) $P\left(\frac{1}{3} < x < \frac{7}{2} \mid x > 1\right)$

$= \frac{P\left[\left(\frac{1}{3} < x < \frac{7}{2}\right) \cap (x > 1)\right]}{P(x > 1)}$

$= \frac{P\left[(x = 1 \text{ or } 2 \text{ or } 3) \cap (x > 1)\right]}{1 - P(X \le 1)}$

$= \frac{P(X = 2 \text{ or } 3)}{1 - P(X = 1)}$

$= \frac{\frac{1}{2}}{1 - \frac{1}{10}} = \frac{1/2}{\frac{10-1}{10}} = \frac{5}{9}$ Ans

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EXPECTATION (Mathematical expectation) –

The average value of a random phenomenon is termed as its mathematical expectation or expected value.

Ex – A gambler might be interested in his average winning at a game. A business man have average profit on a product.)

⇒ The mean of a random variable x i.e. $E(x)$

The expected value of a discrete random variable is a weighted average of all possible value of the random variable where the weights are the probabilities associated with the corresponding values. The mathematical expression for computing the expected values of a discrete random variable x with prob. mass funⁿ $P_x(x)$ is defined as

$$E(x) = \sum_x x \, P_x(x)$$ for discrete random variable

2014 The mathematical expression for computing the expected value of a continuous variable x with prob. distribution funⁿ $F(x)$ is defined as

$$E(x) = \int_{-\infty}^{\infty} x F(x) \, dx$$

Mrginal Probability Function of x :

$$F_X(x) = \int_{-\infty}^{\infty} F_{X,Y}(x,y)\,dy$$ For continous random variable

$$P_Y(y) = \sum_x P_{X,Y}(x,y) = f_Y(y)$$ for discrete random variable

Joint Probability density funⁿ of x and Y

$f_{X,Y}(x,y)$ For continuous random variable

$P_{X,Y}(x,y)$ For discrete random variable

Properties of expectation –

Addition theorum of Expectation : (2017)

Th: If X and Y are random variables then $E(X+Y) = E(X) + E(Y)$ provided all the expectation exist.

Proof

Let X and Y be continuous random variables then by defination

$$E(X) = \int_{-\infty}^{\infty} x\, F_X(x)\,dx \quad \text{—} (1)$$

and

$$E(Y) = \int_{-\infty}^{\infty} y\, F_Y(y)\,dy \quad \text{—} (11)$$

Dipanjan

Now $E(X+Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) F_{X,Y}(x,y)\,dx\,dy$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, F_{XY}(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, F_{XY}(x,y)\,dx\,dy$

$= \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} F_{XY}(x,y)\,dy\right] dx + \int_{-\infty}^{\infty} y \left[\int_{-\infty}^{\infty} F_{XY}(x,y)\,dx\right] dy$

$= \int_{-\infty}^{\infty} x\, F_X(x)\,dx + \int_{-\infty}^{\infty} y\, F_Y(y)\,dy$

$= E(X) + E(Y)$

Multiplication theorem of expectation

Th: If X and Y are independent random variables, then $E(XY) = E(X) \cdot E(Y)$

Proof

let X and Y are continuous random variable with joint probability density funⁿ $F_{XY}(x,y)$ and marginal prob. density functions resp., $F_X(x)$ and $F_Y(y)$ then by defination

$E(X) = \int_{-\infty}^{\infty} x\, F_X(x)\,dx$ and

$E(Y) = \int_{-\infty}^{\infty} y\, F_Y(y)\,dy$

Now, $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, F_{XY}(x,y)\,dy\,dx$

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$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, y\, F_X(x)\, F_Y(y)\, dy\, dx \quad [\because X \text{ and } Y \text{ are independent variables}]$$

$$= \int_{-\infty}^{\infty} x\, F_X(x)\, dx \int_{-\infty}^{\infty} y\, F_Y(y)\, dy$$

$$= E(X) \cdot E(Y)$$

2017

$$E(c) = \int_{-\infty}^{\infty} c \cdot F(x)\, dx = c\int_{-\infty}^{\infty} F(x)\, dx = c \cdot 1 = c$$

$$\sum_x c\, P(X = x) = c\,P(X=1) + c\,P(X=2) + \cdots c\,P(X=n)$$

$$= cP_1 + cP_2 - - - cP_n$$

$$= c\,[P_1 + P_2 - - - + P_n]$$

$$= c \cdot 1$$

$$= c$$

Q) If X is a random variable and a and b are constants then $E(aX+b) = aE(X)+b$ Provided all the expectations exists.

Sol) By defination we have

$$E(aX+b) = \int_{-\infty}^{\infty} (ax+b)\, F(x)\, dx$$

$$= \int_{-\infty}^{\infty} ax\, F(x)\, dx + \int_{-\infty}^{\infty} b\, F(x)\, dx$$

$$= a\int_{-\infty}^{\infty} x\, F(x)\, dx + b\int_{-\infty}^{\infty} F(x)\, dx$$

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$$= a\,E(x) + b\cdot 1$$
$$= a\,E(x) + b$$

★ $\text{Var}(x) = E(x - \bar{x})^2$
$$= E[x - E(x)]^2$$

$$= E(x^2) - [E(x)]^2$$

$\text{Var}(x) = \sigma_x^2 = \frac{1}{N} \sum f_i (x_i - \bar{x})^2$

★ $\text{Var}(x) = E(x - E(x)]^2 = E(x^2) - [E(x)]^2$

Let $E(x) = m$

Now $E[x - E(x)]^2$
$$= \sum_{i=1}^{n} (x_i - m)^2 \, p(x = x_i)$$

$$= \sum_{i=1}^{n} (x_i^2 + m^2 - 2x_i m)\, p_i$$

$$= \sum^{n} x_i^2 p_i + \sum^{n} m^2 p_i - 2 \sum x_i m p_i$$

$$= E(x^2) + m^2 - 2m \sum x_i p_i$$
$$= E(x^2) - m^2$$
$$= E(x^2) - [E(x)]^2$$

or $\text{Var}(x) = E(x - \bar{x})^2$
$$= E(x^2 + \bar{x}^2 - 2x\bar{x})$$
$$= E(x^2) + \bar{x}^2 - 2\bar{x}\bar{x}$$
$$= E(x^2) - \bar{x}^2$$
$$= E(x^2) - [E(x)]^2$$
$$[E(x) = \bar{x}]$$

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Th: If x is a random variable then $Var(ax+b) = a^2\ Var(x)$. where a and b are constants.

Soln

$$\text{Let } Y = ax+b \quad \text{—— (i)}$$

$$\text{then } E(Y) = aE(x)+b \quad \text{—— (ii)}$$

$$Var(Y) = E[Y - E(Y)]^2$$

$$= E[a\{x - E(x)\}]^2$$

$$= a^2 E[x - E(x)]^2$$

$$= a^2\ Var(x)$$

NOTE: $Var(c) = 0$

Q) Let x be a random variable with the following prob. distribution

$X = x$:	-3	6	9
$P(X=x)$:	$1/6$	$1/2$	$1/3$

Find

(i) $E(x)$ (ii) $E(x^2)$ (iii) $E(2x+1)^2$

Soln

(i) $E(x) = \sum x_i p_i = -3\times\frac{1}{6} + 6\times\frac{1}{2} + 9\times\frac{1}{3}$

$$= \frac{-3+18+18}{6} = \frac{33}{6} = \frac{11}{2}$$

ii) $E(x^2) = \sum x_i^2 p_i = 9\times\frac{1}{6} + 36\times\frac{1}{2} + 81\times\frac{1}{3}$

$$= \frac{3}{2} + 18 + 27$$

$$= \frac{3+90}{2} = \frac{93}{2}$$

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iii) $E(2x+1)^2 = E(4x^2+4x+1)$

$= 4E(x^2) + 4E(x) + E(1)$

$= 4 \times \frac{93}{2} + 4 \times \frac{11}{2} + 1$

$= 186 + 22 + 1 = 209$ Ans

Q) $X = x$: -3 -2 -1 0 1 2

$P(X=x)$: 0.07 0.12 0.21 0.25 0.20 0.15

Find

(i) $E(x)$ (ii) $E(x^2)$ (iii) $E(2x+3)$

(iv) $var(x)$ (v) $var(3x+4)$

Soln

(i) $E(x) = \sum x_i p_i = -0.21 - 0.24 - 0.21 + 0.20 + 0.30$

$= -0.66 + 0.50$

$= -0.16$

(ii) $E(x^2) = \sum x_i^2 p_i = 0.63 + 0.48 + 0.21 + 0.20 + 0.60$

$= 2.12$

(iii) $E(2x+3) = 2E(x) + 3$

$= 2 \times (-0.16) + 3$

$= -0.32 + 3$

$= 2.68$

$E[x - E(x)]^2$

(iv) $var(x) = \cancel{[E(x) - E(\bar{x})]^2}$

$= E(x^2) - [E(x)]^2$

$= 2.12 - 0.0256$

$= 2.0944$

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(v) $Var(3x+4) = 3^2 V(x)$
$= 9 \times 2.0944$
$= 18.8496$

Q) Find mean and varience of the prob. distribution funⁿ $F(x)$ of a continuous random variable
$F(x) = 6x(1-x) \quad 0 \le x \le 1$
$= 0 \quad$ elsewhere

Solⁿ
$$E(X) = \int_0^1 x(6x - 6x^2)\,dx$$
$$= \left[\frac{6x^3}{3} - \frac{6x^4}{4}\right]_0^1 = \frac{6}{3} - \frac{6}{4}$$
$$= \frac{24-18}{12} = \frac{1}{2}$$

$$E(X^2) = \int_0^1 x^2(6x - 6x^2)\,dx$$
$$= \int_0^1 (6x^3 - 6x^4)\,dx$$
$$= 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1$$
$$= 6\left[\frac{1}{4} - \frac{1}{5}\right]$$

$$E(X^2) = \frac{6}{20}$$

$$Var(X) = E(X^2) - [E(X)]^2$$
$$= \frac{6}{20} - \frac{1}{4} = \frac{6-5}{20} = \frac{1}{20}$$ Ans

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Q) (I) Find the expectation of the no. on a die when thrown

(II) Two unbiased dice are thrown. Find the expected values of the sum of the numbers on them.

Sol

(I)

$X = x$	1	2	3	4	5	6
$P(X=x)$	1/6	1/6	1/6	1/6	1/6	1/6

Now $E(X) = \sum x_i P(x_i)$

$$= \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6}$$

$$= \frac{21}{6} = \frac{7}{2} = 3.5$$

i.e. 3 or 4

(II)

x	2	3	4	5	6	7	8	9	10	11	12
$P(x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

$E(X) = \sum x_i P(x_i)$

$$= \frac{2+6+12+20+30+42+40+36+30+22+12}{36}$$

$$= \frac{252}{36} = 7 \text{ Ans}$$

Q) An urn contain 7 white and 3 red balls. Two balls are drawn together at random from this urn compute the prob. that neither of them is white. Find also the Prob. of 1 white and 1 red ball hence compute the expected

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no. of white ball drawn.

Soln: Let X be a random variable of getting ~~which~~ two white balls.

Prob of getting no white ball is

$$P(X=0) = \frac{^3C_2}{^{10}C_2} = \frac{3}{\frac{10\times 9}{2}} = \frac{1}{15}$$

Prob. of getting one white ball and one not.

$$P(X=1) = \frac{^3C_1 \times {}^7C_1}{^{10}C_2} = \frac{3\times 7}{\frac{10\times 9}{2}} = \frac{7}{15}$$

Prob. of getting two white ball is

$$P(X=2) = \frac{^7C_2}{^{10}C_2} = \frac{7\times 6}{10\times 9} = \frac{7}{15}$$

12 1/36 We have

$X=x$:	0	1	2
$P(X=x_i)$:	1/15	7/15	7/15

$$\therefore E(X) = \sum_i x\, P(x_i)$$

$$= 0 \cdot \frac{1}{15} + 1 \cdot \frac{7}{15} + 2 \cdot \frac{7}{15}$$

$$= \frac{7+14}{15} = \frac{21}{15} = \frac{7}{5}$$ Ans

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Q) In four tosses of a coin let x be the no. of heads. Calculate the expected value of x.

Sol^n

S = { HHHH, THHH, HTHH, HHTH, HHHT, TTHH, HTTH, HHTT, THTH, HTHT, THHT, TTTH, HTTT, THTT, TTHT, TTTT }

x :	0	1	2	3	4
$P(x)$:	$1/16$	$4/16$	$6/16$	$4/16$	$1/16$

$$\therefore E(X) = \sum x P(x)$$

$$= \frac{4}{16} + \frac{12}{16} + \frac{12}{16} + \frac{4}{16}$$

$$= \frac{1}{4} + \frac{3}{4} + \frac{3}{4} + \frac{1}{4}$$

$$= \frac{8}{4} = 2 \text{ Ans}$$

Q) What is the expectation of the no. of failier preceding the first sucess in an infinite series of independent trail with constant probability p of success in each trail.

Sol^n Let the random variable x denote the no. of failiur preeceding the first sucess Then x can take the values 0, 1, 2 --- ∞ we have

$p(X = x) = q^x p$ (x failures precede the first sucess)

where $q = 1-p$ is the prob. of failure in a trail.

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Then by defination

$$E(X) = \sum_{x=0}^{\infty} x P(x) = \sum_{x=0}^{\infty} x q^x p$$

$$= 0 \cdot q^0 p + 1 \cdot q \cdot p + 2q^2 p + 3 \cdot q^3 p - - -$$

$$= qp(1 + 2q + 3q^2 + 4q^3 + - - -)$$

Let $S = 1 + 2q + 3q^2 + 4q^3 + - - -$

$qS = q + 2q^2 + 3q^3 + 4q^4 + - - -$

Subtracting we get

$$(1-q)S = 1 + q + q^2 + q^3 + - - -$$

$$(1-q)S = \frac{1}{1-q}$$

$$S = \frac{1}{(1-q)^2}$$

$$E(X) = pq \frac{1}{(1-q)^2}$$

$$= pq \cdot \frac{1}{p^2}$$

$$E(X) = \frac{q}{p}$$

Q) A coin is tossed until a head appears. what is the expectation of the number of tosses required.

Soln Let x denote the no. of tosses required to get the first head then

Event	x	Probability $P(x)$
H	1	$1/2$
TH	2	$1/2 \times 1/2 = 1/4$
TTH	3	$1/2 \times 1/2 \times 1/2 = \frac{1}{8}$
TTTH	4	$1/2 + 1/2 \times 1/2 \times 1/2 = 1/16$
⋮	⋮	⋮

$\therefore E(X) = \sum_{x=1}^{\infty} x_i P(x_i) = 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{8}$

$+ 4 \times \frac{1}{16} + \cdots$

Let $S = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \cdots$

then $\frac{1}{2} S = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{16} + \cdots$

$\therefore \left(1 - \frac{1}{2}\right) S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

$\frac{1}{2} S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

$= \frac{1/2}{1 - \frac{1}{2}} = 1$

$\Rightarrow S = 2$

$\therefore E(X) = 2$

Hence expected no. of tosses are 2.

v.v.g

4 Q) A box contains 'a' white and 'b' black balls c balls are drawn at random. Find the expected value of the no. of white balls drawn.

Solⁿ let a variable x_i associated with i^{th} draw be defined as follows

$x_i = \begin{cases} 1 & \text{if } i^{th} \text{ ball drawn is white} \\ 0 & \text{if } i^{th} \text{ ball drawn is black} \end{cases}$

$E(x_i) = (x_i = 0) \cdot P(x_i = 0) + (x_i = 1) \, P(x_i = 1)$

$= 0 \cdot \frac{b}{a+b} + 1 \cdot \frac{a}{a+b}$

$$\Rightarrow E(X_i) = \frac{a}{a+b}$$

Now the no.s of the white balls among 'c' balls drawn is given by

$$S = X_1 + X_2 + X_3 - - - X_c$$

$$\Rightarrow E(S) = \sum_{i=1}^{c} E(X_i)$$

$$= \sum_{i=1}^{c} \frac{a}{a+b} = \frac{ac}{a+b}$$ Ans

Moment Generating Function (M.g.F.) —

The moment generating funⁿ of a random variable x (about origin) having the probability funⁿ $f(x)$ is given by

$$M_x(t) = E[e^{tx}] = \begin{cases} \int e^{tx} f(x)\,dx & \text{for continuous prob. distribution} \\ \sum e^{tx} f(x) & \text{for discrete prob. distri.} \end{cases}$$

$$M_x(t) = E[e^{tx}]$$

$$= E\left[1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + - - \frac{t^r x^r}{r!} + - -\right]$$

$$= \left[1 + tE(x) + \frac{t^2}{2!}E(x^2) + - - \frac{t^r}{r!}E(x^r) + - - \right]$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + - - \frac{t^r}{r!}\mu_r' - - \quad (1)$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r' \quad \text{where } E(x^r) = \mu_r'$$

~~E(x^r) = \mu_r'~~

$E(x$

$E(X^r) = \mu_r'$

Diff. w.r.t. t of eq^n ① we get

$$\frac{dM_x(t)}{dt} = \mu_1' + \frac{2t\mu_2'}{2!} + \frac{3t^2\mu_3'}{3!} + \cdots + \frac{rt^{r-1}}{r!}\mu_r' - - -$$

$$\frac{d^2M_x(t)}{dt^2} = \frac{2!\mu_2'}{2!} + \frac{3.2t}{3!}\mu_3' + \frac{4.3t^2}{4!}\mu_4' + \cdots$$

$$\frac{d^3M_x(t)}{dt^3} = \frac{3!\,\mu_3'}{3!} + \frac{4.3.2\,t\,\mu_4'}{4!} + \frac{5.4.3.t^2\mu_5'}{5!} + \cdots$$

$$= \frac{3!}{3!}\mu_3' + \frac{4!}{4!}t\mu_4' + \frac{5!}{5!}\frac{t^2}{2}\mu_5' + \cdots$$

$$\frac{d^rM_x(t)}{dt^r} = \frac{r!}{r!}\mu_r' + \frac{(r+1)!}{(r+1)!}t\mu_{r+1}' + \frac{(r+2)!}{(r+2)!}\frac{t^2\mu_{r+2}'}{2}$$

$$= \mu_r' + t\mu_{r+1}' + \frac{t^2}{2}\mu_{r+2}' + \cdots$$

$$\Rightarrow \left[\frac{d^rM_x(t)}{dt^r}\right]_{t=0} = \mu_r'$$

Q) Let the random variables x assume the values r with the prob. law
$P(X=r) = q^{r-1}p,\ r = 1, 2, 3 - - -$
Find the moment generating fun^n of x and hence its mean and variance.

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Sol^n $M_x(t) = \sum e^{tx} F(x)$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (e^t q)^x$$

$$= \frac{p}{q} \left\{ e^t q + (e^t q)^2 + (e^t q)^3 + \cdots \right\}$$

$$= \frac{p}{q} e^t q \left\{ 1 + e^t q + (e^t q)^2 + \cdots \right\}$$

$$= pe^t \left\{ \frac{1}{1 - e^t q} \right\}$$

$$\Rightarrow M_x(t) = pe^t \left(\frac{1}{1 - qe^t} \right)$$

$$\frac{d M_x(t)}{dt} = \frac{(1 - qe^t) pe^t + pe^t qe^t}{(1 - qe^t)^2}$$

$$= \frac{pe^t}{(1 - qe^t)^2}$$

$$\left[\frac{d M_x(t)}{dt} \right]_{t=0} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$\frac{d^2 M_x(t)}{dt^2} = \frac{(1-qe^t)^2 pe^t + pe^t 2(1-qe^t) . qe^t}{(1 - qe^t)^4}$$

$$= \frac{pe^t - pqe^{2t} + 2pqe^{2t}}{(1 - qe^t)^3}$$

$$= \frac{pe^t + pqe^{2t}}{(1 - qe^t)^3}$$

$$\left[\frac{d^2 M_x(t)}{dt^2}\right]_{t=0} = \frac{p + pq}{(1-q)^3} = \frac{p(1+q)}{p^3} = \frac{1+q}{p^2}$$

$$\therefore \mu_2' = \frac{1+q}{p^2} - \frac{1}{p^2}$$

$$= \frac{q}{p^2} \quad \text{Ans}$$

Q) If the moment of varient x are defined by

$E(x^r) = 0.6$, $r = 1, 2, 3 - - -$

Show that $P(x=0) = 0.4$, $P(x=1) = 0.6$, $P(x \geq 2) = 0$

Sol^n $M_x(t) = E(e^{tx})$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$$

$$= 1 + \sum_{r=1}^{\infty} \frac{t^r}{r!} \mu_r'$$

$$= 1 + \sum_{r=1}^{\infty} \frac{t^r}{r!} (0.6)$$

$$= 1 + \sum_{r=1}^{\infty} \frac{t^r}{r!} (0.6) + 0.6 - 0.6$$

$$= 0.4 + \sum_{r=1}^{\infty} \frac{t^r}{r!} (0.6) + 0.6$$

$$= 0.4 + \left[\sum_{r=0}^{\infty} \frac{t^r}{r!} (0.6)\right]$$

$$M_x(t) = 0.4 + 0.6 e^t \quad \text{——— (1)}$$

also

$$M_x(t) = E(e^{tx})$$

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$$= \sum_{x=0}^{\infty} e^{tx} \; P(X=x)$$

$$= e^{t\cdot 0} P(X=0) + e^{t} P(X=1) + e^{2t} P(X=2) + \cdots$$

$$= P(X=0) + e^{t} P(X=1) + \sum_{x=2}^{\infty} e^{tx} P(X=2) \quad \text{— (ii)}$$

From (i) & (ii)

$P(X=0) = 0.4$

$P(X=1) = 0.6$

$P(X \geq 2) = 0$

Proved

Q) Find the moment generating fun^n of the random variable whose moments are
$\mu_r' = (r+1)!\, 2^r$

Sol^n. $M_x(t) = E[e^{tx}]$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} (r+1)!\, 2^r$$

$$= \sum_{r=0}^{\infty} t^r (r+1)\, 2^r$$

$$= \sum_{r=0}^{\infty} (2t)^r (r+1)$$

$$= 1 + 2(2t) + 3(2t)^2 + 4(2t)^3 + \cdots$$

Let $S = 1 + 2m + 3m^2 + 4m^3 + \cdots$ where $m = 2t$

$rS = m + 2m^2 + 3m^3 + \cdots$

$(1-m) S = 1 + m + m^2 + m^3 + \cdots$

$(1-m) S = \frac{1}{(1-m)}$

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$$\Rightarrow S = \frac{1}{(1-m)^2}$$

$$= \frac{1}{(1-2t)^2} = (1-2t)^{-2}$$

Properties of moment generating funⁿ –

1. $M_{cx}(t) = M_x(ct)$ c being a constant

$$M_{cx}(t) = E[e^{t(cx)}]$$
$$= E[e^{(ct)x}]$$
$$= M_x(ct)$$

2. The m.g.f of the sum of a no. of independent random variables is equal to the product of their respective moment generating funⁿ.

Symbolically if $x_1, x_2, x_3 \dots x_n$ are independent random variable then the moment generating funⁿ of their sum $x_1 + x_2 + x_3 \dots + x_n$ is given by

$$M_{x_1+x_2+x_3 \dots x_n}(t) = M_{x_1}(t) . M_{x_2}(t) . M_{x_3}(t) \dots \dots M_{x_n}(t)$$

$$M_{x_1+x_2+x_3 \dots x_n}(t) = E[e^{t(x_1+x_2+x_3 \dots x_n)}]$$

$$= E[e^{tx_1} . e^{tx_2} \dots e^{tx_n}]$$

$$= E(e^{tx_1}) . E(e^{tx_2}) \dots E(e^{tx_n})$$
$$= M_{x_1}(t) . M_{x_2}(t) \dots M_{x_n}(t)$$

iii) Effect of change of origin and scale on m.g.f.

Let us transform x to the new variable U by changing both the origin and scale in x as follows $U = \frac{x-a}{h}$ where a and h are constants.

Now m.g.f of U about origin is given by

$$M_U(t) = E[e^{tU}]$$

$$= E\left[e^{t\left(\frac{x-a}{h}\right)}\right]$$

$$= e^{-\frac{at}{h}} E\left(e^{(t/h)x}\right)$$

$$= e^{\frac{at}{h}} M_x(t/h)$$

where $M_x(t)$ is the m.g.f of x about origin.

$$\boxed{\begin{aligned} &\text{If } z = \frac{x-\mu}{\sigma} \\ &M_z(t) = e^{-\mu t/\sigma} M_x\left(\frac{t}{\sigma}\right) \end{aligned}}$$

iv) Uniqueness theorem of m.g.f –

The moment generating funⁿ of a distribution if it exist, uniquely determines the distribution this implies that the corresponding to a given prob. distribution there is only one m.g.f (provided it exist) and corresponding to a given m.g.f there is only one prob. distribution.

Hence $M_x(t) = M_y(t)$

$\Rightarrow$ X and Y are identically distributed.

Binomial Distribution

A random variable x is said to follow binomial distribution if it assumes only non-ve values and its p.m.f. is given by

$$P(X=x) = {}^{n}C_{x}\, p^{x} q^{n-x} \quad x = 0, 1, 2, \ldots n \text{ and } q = 1-p \qquad \text{(1)}$$
$$= 0 \quad \text{otherwise}$$

Where the two independent constants n and p in the distribution are known as the parameter of the distribution. n is also known as the degree of binomial distribution. It is also denoted as $b(x, n, p)$

Mean of binomial distribution –

$$E(X) = \mu_1'$$

$$\mu_1' = E(X) = \sum x\, P(x)$$

$$= \sum_{x=0}^{n} x\, {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$= \sum_{x=1}^{n} x \cdot \frac{n}{x}\, {}^{n-1}C_{x-1}\, p^{x} q^{n-x}$$

$$= np \sum_{x=1}^{n} {}^{n-1}C_{x-1}\, p^{x-1} q^{n-1-(x-1)}$$

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$$= np\left[{}^{n-1}C_0\, p^0 q^{n-1} + {}^{n-1}C_1\, p^1 q^{n-2} + \cdots\right]$$

$$= np\,[p+q]^{n-1} = np \qquad [\because q = 1-p]$$

(1)

$$\mu_2' = E(x^2) = \sum x^2 P(x)$$

$$= \sum x^2\, {}^nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} [x(x-1)+x]\; {}^nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)\; {}^nC_x\, p^x q^{n-x} + np \;(\text{from } 1)$$

$$= \sum_{x=2}^{n} x(x-1)\,\frac{n(n-1)}{x(x-1)}\; {}^{n-2}C_{x-2}\, p^x q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2}\, p^{x-2} q^{n-x} + np$$

$$= n(n-1)\,p^2\,(p+q)^{n-2} + np$$

$$= n(n-1)\,p^2 + np$$

$$= n^2p^2 - np^2 + np$$

Variance –

$$\mu_2 \text{ (varience)} = \mu_2' - (\mu_1')^2$$

$$= n^2p^2 - np^2 + np - (np)^2$$

$$= np - np^2$$

$$= np\,(1-p)$$

$$= npq$$

M.g.f

$$M_x(t) = E[e^{tx}]$$

$$= \sum_{x=0}^{\infty} e^{tx}\, P(X = x)$$

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$$= e^{t0} P(x=0) + e^{t} P(x=1) + e^{2t} P(x=2) + \ldots$$
$$\ldots + e^{nt} P(x=n)$$

$$= e^{t0}\, {}^nC_0 p^0 q^n + e^t\, {}^nC_1 p q^{n-1} + e^{2t}\, {}^nC_2 p^2 q^{n-2}$$
$$+ \ldots\ e^{nt}\, {}^nC_n p^n q^0$$

$$= {}^nC_0 (e^t p)^0 q^n + {}^nC_1 (pe^t) q^{n-1} + {}^nC_2 (pe^t)^2 q^{n-2}$$
$$+ \ldots\ {}^nC_n (e^t p)^n q^0$$

$$= (q + pe^t)^n$$

Q) The mean and variance of a binomial distribution are 4 and 3 respectively. Find the parameter n and p.

Sol^n Given mean of binomial distribution is

$np = 4$ —— (I)

and varience is

$npq = 3$ —— (II)

From (I) & (II) we get

$$q = \frac{3}{4}$$

$$\Rightarrow p = 1 - \frac{3}{4} = \frac{1}{4}$$

$$n \cdot \frac{1}{4} = 4$$

$$\Rightarrow n = 16$$

$$\therefore p = \frac{1}{4} \ \& \ n = 16$$

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Mode of binomial distribution –

Mode of the distribution is that value of the random variable for which the probability is maximum.

Let $P(X = x)$ is max^m

$P(X = x-1) \leq P(X = x) \geq P(X = x+1)$

$$^nC_{x-1}\, p^{x-1} q^{n-x+1} \leq {}^nC_x\, p^x q^{n-x} \geq {}^nC_{x+1}\, p^{x+1} q^{n-x-1}$$

Dividing by $^nC_x\, p^x q^{n-x}$

$$\frac{x}{n-x+1}\, p^{-1} q^{1} \leq 1 \geq \frac{n-x}{(x+1)}\, p q^{-1}$$

$$\frac{x}{n-x+1}\, \frac{q}{p} \leq 1 \geq \frac{n-x}{x+1}\, \frac{p}{q}$$

From first two inequality

$$xq \leq (n-x+1)p$$

$$x(q+p) \leq (n+1)p$$

$$x \leq \frac{(n+1)p}{(p+q)} = (n+1)p$$

From last two inequality

$$q(x+1) \geq (n-x)p$$

$$qx+q \geq np - xp$$

$$x(p+q) \geq np-q$$

$$x \geq \frac{np-q}{p+q} = \frac{np-1+p}{p+q}$$

$$= p(n+1)-1$$

Case 1 –

when $(n+1)p - 1 = x$ (an integer)